

# BERNOULLI THEOREM GENERALIZED TO RHEOLOGICALLY COMPLEX VISCOUS FLUID FLOW

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*The Bernoulli theorem is generalized to two-dimensional and axisymmetric micropolar incompressible fluid flows. It is shown that the approach developed is also applicable to magnetohydrodynamic flows of a viscous Newtonian fluid.*

It is a known fact in classical hydrodynamics that the ideal incompressible fluid flow obeys, at certain conditions, the Bernoulli equation [1]. Generalization of this equation to the case of potential flows of an ideal gas is also well known [2]. In connection with this, other generalizations are of interest, for instance, generalizations to viscous fluid flows and rheologically complex fluid flows. Different generalizations of the Bernoulli theorem to a viscous incompressible fluid are obtained in [3-5].

The present work is aimed at a generalization of the results [3, 4] to two-dimensional flows of one of the popular models of rheologically complex fluids, i.e., a micropolar fluid. A hydrodynamic description of the micropolar fluid requires the introduction of an additional macroscopic vector variable  $\Omega$ , which is interpreted as the angular speed of microrotation. We note the known mathematical similarity of the micropolar fluid with magnetic hydrodynamics (MHD). Indeed, in order to describe MHD, it is also necessary to introduce an additional vector field, i.e. a magnetic field  $\mathbf{H}$ ; here the mathematical properties of the vectors  $\Omega$  and  $\mathbf{H}$  are similar since they are axial. In light of this, it is not surprising that the Bernoulli theorem may be generalized to MHD flows as well. The corresponding results are described in briefly at the end of this paper.

Now we move to the derivation of the Bernoulli theorem's analogue for a micropolar incompressible fluid. For this, we need an equation describing the evolution of the velocity  $\mathbf{V}$  field. In all the variants of the micropolar theory [6-8] it has one and the same form

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\omega} \times \mathbf{V} \right) + (\mu + k) \text{rot } \boldsymbol{\omega} - k \text{rot } \Omega + \nabla B = 0. \quad (1)$$

Here  $\boldsymbol{\omega} = \text{rot } \mathbf{V}$  is the vorticity;  $\rho$  the density;  $\mu$  the dynamic viscosity coefficient;  $k$  the coefficient for particle cohesion with its environment. The Bernoulli function  $B$  is given by the conventional expression

$$B = \frac{\rho V^2}{2} + p, \quad (2)$$

where  $p$  is the pressure. For two-dimensional vectors, the vectors  $\boldsymbol{\omega}$  and  $\Omega$  have only one nonzero component

$$\boldsymbol{\omega} = (0, 0, \omega), \quad \Omega = (0, 0, \Omega). \quad (3)$$

An important property of the similar vector fields is the possibility of introducing the additional vectors  $\mathbf{a}$  and  $\mathbf{A}$  according to the rule

$$\text{rot } \boldsymbol{\omega} = -\boldsymbol{\omega} \times \mathbf{a}, \quad \mathbf{a} = \nabla \ln \omega, \quad (4)$$

$$\text{rot } \Omega = -\Omega \times \mathbf{A}, \quad \mathbf{A} = \nabla \ln \Omega. \quad (5)$$

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In the case of steady-state flows, Eq. (1), taking into account (4) and (5), may be rewritten in the form

$$\boldsymbol{\omega} \times \mathbf{U} + \nabla B = 0, \quad (6)$$

where

$$\mathbf{U} = \rho \mathbf{V} - (\mu + k) \mathbf{a} + k \frac{\boldsymbol{\Omega}}{\omega} \mathbf{A}. \quad (7)$$

From Eq. (6) it follows that

$$\mathbf{U} \times \nabla B = 0. \quad (8)$$

Thus, we pass to the next formulation of the Bernoulli theorem for the steady-state micropolar fluid flows: the Bernoulli function  $B$  remains constant along the flow lines of the vector field  $\mathbf{U}$ .

Generalization of the formulated Bernoulli theorem to axisymmetric flows appears to be similar to the two-dimensional case. Therefore, we give here only the final expression for the vector field  $\mathbf{U} = (U_r, U_\theta, 0)$  in the spherical coordinate system  $(r, \theta, \varphi)$ :

$$U_r = \rho V_r - \frac{\mu + k}{\omega r} \frac{\partial}{\partial r} (\omega r) + \frac{k}{\omega r} \frac{\partial}{\partial r} (\Omega r), \quad (9)$$

$$U_\theta = \rho V_\theta - \frac{\mu + k}{\omega r \sin \theta} \frac{\partial}{\partial \theta} (\omega \sin \theta) + \frac{k}{\omega r \sin \theta} \frac{\partial}{\partial \theta} (\Omega \sin \theta). \quad (10)$$

In the limiting case  $k = 0$ , the results of (7)-(10) pass into those of [3, 4].

Note that in the limit of small Reynolds numbers the Bernoulli function turns into a pressure term,  $B = p$ , and, therefore, the flow lines of the vector field  $\mathbf{U}$  are isobars for Stokes flows. This result is easy to check for exactly solvable problems, for instance, for the problem on flow around a sphere, by performing explicit calculations [9].

Next, we extend the Bernoulli theorem in a manner similar to MHD flows of a viscous Newtonian fluid. The velocity  $\mathbf{V}$  field in this case is described by the following equation [10]:

$$\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\omega} \times \mathbf{V} \right) + \mu \operatorname{rot} \boldsymbol{\omega} + \nabla B + \frac{1}{4\pi} (\mathbf{H} \times \operatorname{rot} \mathbf{H}) = 0. \quad (11)$$

Now we choose the vector field  $\mathbf{A}_H$  satisfying the relation

$$\operatorname{rot} \boldsymbol{\omega} + \frac{1}{4\pi\mu} \mathbf{H} \times \operatorname{rot} \mathbf{H} = -\boldsymbol{\Omega} \times \mathbf{A}_H. \quad (12)$$

It is easy to ensure that in the case of two-dimensional flows  $\mathbf{V} = (V_1, V_2, 0)$  and  $\mathbf{H} = (H_1, H_2, 0)$ , the vector  $\mathbf{A}_H$  is given by the expression

$$\mathbf{A}_H = \nabla \ln \left( \frac{\omega}{h_3} \right) + \frac{J}{c\mu\omega} \mathbf{H}. \quad (13)$$

Here  $h_3$  is the Lamé coefficient equal to 1 for two-dimensional flows and  $r$  for axisymmetric flows (where  $r$  is the distance from the axis of symmetry). In terms of  $J$ , we designate a single nontrivial component of current density  $\mathbf{j}$ :

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{rot} \mathbf{H} = (0, 0, J), \quad (14)$$

$$J = \frac{c}{4\pi} h_1 h_2 \left[ \frac{\partial}{\partial q_1} \left( \frac{H_2}{h_2} \right) - \frac{\partial}{\partial q_2} \left( \frac{H_1}{h_1} \right) \right], \quad (15)$$

where  $q_1$  and  $q_2$  are the generalized coordinates.

On the whole, the formulation of the Bernoulli theorem almost does not change as compared to the case of a micropolar fluid: the Bernoulli function  $B$  remains constant along the flow lines of the vector field  $\mathbf{U}$  where

$$\mathbf{U} = \rho \mathbf{V} - \mu \mathbf{A}_H. \quad (16)$$

One may verify the validity of the result obtained by performing direct calculations in exactly solvable problems, e.g., in the Hartmann problem on MHD flow between parallel planes [10].

Finally, it is worth noting that the results obtained may be useful, in addition to being of general interest, for checking the accuracy of numerical results of steady-state MHD and micropolar fluid flows.

#### NOTATION

$\Omega$ , angular speed of microrotation;  $\mathbf{H}$ , magnetic field;  $\mathbf{V}$ , velocity;  $B$ , Bernoulli function;  $\mathbf{a}$ ,  $\mathbf{A}$ ,  $\mathbf{A}_H$ , additional vector fields determined by formulas (4), (5), and (13), respectively;  $(r, \theta, \varphi)$ , spherical coefficients;  $\mathbf{J}$ , current density;  $c$ , light velocity;  $h_1, h_2, h_3$ , Lamé coefficients.

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